

# RECURSION OPERATOR FOR A SYSTEM ADMITTING LAX REPRESENTATION WITH NON-RATIONAL LAX FUNCTION

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## **Abstract**

A recursion operator is constructed for a hydrodynamic type system admitting dispersionless Lax representation with non-rational Lax function.

# 1 Introduction

In the present paper we construct a recursion operator for a hydrodynamic type system admitting Lax representation with a non-rational Lax function. Let us note that in the case of rational and polynomial Lax functions we can construct a recursion operator following [1]. We write a recursion relation

$$L_{t_{n+1}} = LL_{t_n} + \{R_n, L\} \quad (1)$$

between the symmetries. In case of a polynomial or rational Lax function the form of the remainder  $R_n$  can be predicted. So, we can use the above recursion relation to find a recursion operator, see [1]-[4] for detail. For the non-rational Lax function it is not possible to predict the form of the remainder  $R_n$  and apply the method of [1]. So we construct the recursion operator analysing the Lax equation itself. For constructions of recursion operators of some other classes of hydrodynamic type systems see also [5]-[7].

Let us give necessary definitions. We introduce the algebra of Laurent series

$$\mathcal{A} = \left\{ \sum_{-\infty}^{\infty} u_i p^i : u_i \text{ -- smooth rapidly decreasing at infinity functions} \right\}, \quad (2)$$

with the Poisson bracket given by

$$\{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p}. \quad (3)$$

Using the Gelfond-Dikkii construction [8] we can write the hierarchy of integrable equations on the algebra  $\mathcal{A}$ . Equations of such type have applications in the topological field theories [9] and appear as equations describing slow varying quasi periodic solutions of non-linear integrable equations, see [10]-[17]. We consider non-rational Lax function

$$L = \mu - m \ln(\mu - c^1) + \ln(\mu - c^2) + \cdots + \ln(\mu - c^{m+1}). \quad (4)$$

The corresponding Lax equation

$$L_t = \{(L^2)_{\geq 0}, L\} \quad (5)$$

leads to the system

$$c_t^j = \partial_x \left( \frac{(c^j)^2}{2} + mc^1 - c^2 - \dots - c^{m+1} \right), \quad (6)$$

where  $j = 1, 2, \dots, (m+1)$ , see [18],[19] and references there in. We have a hierarchy of symmetries for the above equation given by

$$L_t = \{(L^n)_{\geq 0}, L\} \quad n = 1, 2, \dots \quad (7)$$

As we show the above hierarchy admits the following recursion operator

$$\mathcal{R} = A\partial_x^{-1}, \quad (8)$$

where matrix  $A = (\gamma_{ij})$  has entries

$$\begin{aligned} \gamma_{11} &= c_x^1 + \sum_{j=2}^{m+1} \frac{c_x^1 - c_x^j}{c^1 - c^j}, & \gamma_{1k} &= -\frac{c_x^1 - c_x^k}{c^1 - c^k}, & \gamma_{k1} &= m \frac{c_x^1 - c_x^k}{c^1 - c^k}, \\ \gamma_{kk} &= c_x^k - m \frac{c_x^1 - c_x^k}{c^1 - c^k} + \sum_{j=2, j \neq k}^{m+1} \frac{c_x^k - c_x^j}{c^k - c^j}, & \gamma_{ki} &= -\frac{c_x^k - c_x^i}{c^k - c^i} \end{aligned}$$

$k \neq i$ , and  $k, i = 2, 3, \dots, m+1$ .

The paper is organized as follows. In Section 2 we give general construction of the recursion operator and Section 3 we consider several examples.

## 2 Evaluation of recursion operator

Let us introduce new variables

$$c^1 = u \quad \text{and} \quad v^{j-1} = c^1 - c^j \quad j = 2, 3, \dots, (m+1). \quad (9)$$

In new variables the system (6) takes the form

$$\begin{aligned}
u_t &= uu_x + v_x^1 + \dots v_x^n \\
v_t^1 &= v^1 u_x + (u - v^1) v_x^1 \\
&\dots \\
v_t^m &= v^m u_x + (u - v^m) v_x^m
\end{aligned} \tag{10}$$

The system (10) admits a Lax representation

$$L_t = \{(L^2)_{\geq 1}, L\} \tag{11}$$

with Lax function

$$L = p + u + \ln \left( 1 + \frac{v^1}{p} \right) + \ln \left( 1 + \frac{v^2}{p} \right) + \dots + \ln \left( 1 + \frac{v^m}{p} \right). \tag{12}$$

Thus we have the whole hierarchy of symmetries for the system (10) given by

$$L_{t_n} = \{(L^n)_{\geq 1}, L\} \quad n = 1, 2, \dots \tag{13}$$

Let us construct a recursion operator for the above hierarchy of symmetries. We construct the recursion operator by direct analysis of the Lax representation. Let

$$L^n = a_n p^n + a_{n-1} p^{n-1} + \dots a_1 p + a_0 + a_{-1} p^{-1} + \dots \tag{14}$$

The next two lemmas give some relations between coefficients of  $L^n$  and

$$L_{t_n} = u_{t_n} + \frac{v_{t_n}^1}{p+1} + \dots + \frac{v_{t_n}^m}{p+1}.$$

**Lemma 2.1** *For any  $k = 2, 3 \dots m$  and any  $n = 2, 3, \dots$  the following equality holds*

$$\sum_{i=1}^n (-1)^{(i-1)} a_i (v^k)^i = \partial_x^{-1} v_{t_n}^k \tag{15}$$

**Proof.** Using (14) we can write the equation (13) as

$$\begin{aligned} u_{t_n} + \frac{v_{t_n}^1}{p+v^1} + \cdots + \frac{v_{t_n}^m}{p+v^m} = \\ (na_n p^{n-1} + \cdots + 2a_2 p + a_1) \left( u_x + \frac{v_x^1}{p+v^1} + \cdots + \frac{v_x^m}{p+v^m} \right) - \\ (a_{n,x} p^n + \cdots + a_{2,x} p^2 + a_{1,x}) \left( 1 - \frac{v^1}{p(p+v^1)} - \cdots - \frac{v^m}{p(p+v^m)} \right) \end{aligned}$$

Multiplying the above equation by  $(p+v_1)(p+v_2)\dots(p+v_m)$  and then substituting  $p = -v_k$  we obtain

$$v_{t_n}^k = \sum_{i=1}^n (-1)^{i-1} i a_i (v^k)^{i-1} v_x^k + \sum_{i=1}^n (-1)^{i-1} a_{i,x} (v^k)^i.$$

That is

$$v_{t_n}^k = \left( \sum_{i=1}^n (-1)^{i-1} i a_i (v^k)^{i-1} v_x^k + \sum_{i=1}^n (-1)^{i-1} a_{i,x} (v^k)^i \right)_x. \quad \square$$

**Lemma 2.2** *For any  $n = 2, 3, \dots$  the following equality holds  $a_0 = \partial_x^{-1} u_{t_n}$ .*

**Proof.** The Lax equation (13) can be written as

$$L_{t_n} = \{(L^n)_{\leq 0}, L\} \quad n = 1, 2, \dots$$

Using (14) and collecting coefficients of zero power of  $p$  in the above equations we have  $u_{t_n} = a_{0,x}$ .  $\square$

The above lemmas allow us to express the coefficients of  $(L_{>0}^{(n+1)})_p$  and  $(L_{>0}^{(n+1)})_x$  in terms of coefficients of  $L_{\geq 0}^n$  and  $L_{t_n}$ .

**Lemma 2.3** *Let*

$$\frac{1}{n+1} \left( L_{\geq 1}^{(n+1)} \right)_p = b_n p^{n-1} + \cdots + b_2 p + b_1. \quad (16)$$

*Then*

$$b_r = a_{r-1} + \sum_{k=1}^m \sum_{j=0}^{r-1} (v^k)^{-j} a_{r-j} + \sum_{k=1}^m (v^k)^{-r} \partial_x^{-1} v^k, \quad (17)$$

*where  $r = 1, 2, \dots, m$ .*

Let

$$\frac{1}{n+1} \left( L_{\geq 1}^{(n+1)} \right)_x = d_n p^n + \cdots + d_2 p^2 + d_1 p. \quad (18)$$

Then

$$d_r = u_x a_r + \sum_{k=1}^m \sum_{j=0}^{r-1} (v^k)^{-j-1} v_x^k a_{r-j} + \sum_{k=1}^m (v^k)^{-r-1} v_x^k \partial_x^{-1} v^k, \quad (19)$$

where  $r = 1, 2, \dots, m$ .

**Proof.** We have

$$\frac{1}{n+1} \left( L_{\geq 1}^{(n+1)} \right)_p = \left( L_{\geq 0}^{(n)} L_p \right)_{\geq 0}.$$

That is

$$\frac{1}{n+1} \left( L_{\geq 1}^{(n+1)} \right)_p = \left( (a_n p^n + \cdots + a_0) \left( u_x + \sum_{k=1}^m \frac{v_x^k}{p + v_k} \right) \right)_{\geq 0}.$$

For each  $k = 1, \dots, m$ , we expand  $\frac{1}{p + v^k}$  as series in terms of  $p^{-1}$  around  $p = \infty$  and multiply with  $(a_n p^n + \cdots + a_0)$ . Collecting coefficients of  $p^k$ ,  $k = 1, \dots, m$ , in the above equality and using Lemma 2.1 we obtain formula (17). The formula (19) is obtained in the same way.  $\square$

Using the above lemmas we find a recursion operator for the hierarchy (13).

**Lemma 2.4** *The recursion operator for the system (10) can be written as  $\mathcal{R} = C \partial_x^{-1}$ , where  $C$  is an  $(m+1) \times (m+1)$  matrix. It is convenient to write the matrix  $C$  as a sum of two matrices,  $C = (A + B)$ . The matrix  $A = (\alpha_{ij})$  has entries*

$$\alpha_{11} = u_x;$$

$$\alpha_{1(j+1)} = v_x^j (v^j)^{-1}, \quad j = 1, 2, \dots, m;$$

$$\alpha_{(j+1)1} = v_x^j, \quad j = 1, 2, \dots, m;$$

$$\alpha_{(j+1)(j+1)} = (u_x - v_x^j), \quad j = 1, 2, \dots, m;$$

$$\alpha_{(i+1)(j+1)} = 0 \quad i \neq j \quad i, j = 1, 2, \dots, m;$$

The matrix  $B = (\beta_{ij})$  has entries

$$\begin{aligned}
\beta_{11} &= 0; \\
\beta_{1(j+1)} &= 0, \quad j = 1, 2, \dots, m; \\
\beta_{(j+1)1} &= 0, \quad j = 1, 2, \dots, m; \\
\beta_{(j+1)(j+1)} &= \sum_{k=1, k \neq j}^m \frac{v_x^k - v^k(v^j)_x(v^j)^{-1}}{v^k - v^j} \quad j = 1, 2, \dots, m; \\
\beta_{(i+1)(j+1)} &= \frac{v_x^i - v^i v_x^j (v^j)^{-1}}{v^j - v^i} \quad i \neq j \quad i, j = 1, 2, \dots, m;
\end{aligned}$$

**Proof.** Using notations of Lemma 2.3 the Lax equation (13) can be written as

$$\begin{aligned}
u_{t_{n+1}} + \sum_{k=1}^m \frac{v_{t_{n+1}}^k}{p + v^k} = \\
(n+1)(b_n p^{n-1} + \dots + b_2 p + b_1) \left( u_x + \sum_{k=1}^m \frac{v_x^k}{p + v^k} \right) - \\
(n+1)(d_n p^n + \dots + d_2 p^2 + d_1) \left( 1 - \sum_{k=1}^m \frac{v^k}{p(p + v^k)} \right).
\end{aligned} \tag{20}$$

We multiply the above equation by  $(p + v^1)(p + v^2) \dots (p + v^m)$  and substitute expressions for  $b_i, d_i$ ,  $i = 1, 2, \dots, n$ , given in Lemma 2.3. Equating coefficients of  $p^k$ ,  $k = 1, 2, \dots, m$ , we obtain a system of equations linear with respect to  $v_{t_{n+1}}^k$ ,  $k = 1, 2, \dots, m$ . Solving the system we obtain the recursion operator given above.  $\square$

**Remark 2.5** Let us define vector  $V = (u, v^1, v^2, \dots, v^m)$  and write the system (10) as

$$V_t = K(V, V_x). \tag{21}$$

It follows by direct calculations that the constructed above operator satisfies the criteria for recursion operators

$$\mathcal{R}_t = \mathbb{D}_K \mathcal{R} - \mathcal{R} \mathbb{D}_K, \tag{22}$$

where  $\mathbb{D}_K$  is the Freshet derivative of  $K$ .

Returning to the original variables  $c^1, \dots, c^{m+1}$  we obtain the recursion operator (8).

### 3 Examples

Let us consider some examples. We give examples in variables  $c^1, c^2, \dots, c^{m+1}$ .

**Example 1** *Let us consider equation (6) with  $m = 1$ . The equation becomes*

$$\begin{aligned} c_t^1 &= c^1 c_x^1 + c_x^1 - c_x^2 \\ c_t^2 &= c^2 c_x^2 + c_x^1 - c_x^2 \end{aligned} \tag{23}$$

*The above system admits the recursion operator*

$$\begin{pmatrix} c_x^1 + \frac{c_x^1 - c_x^2}{c^1 - c^2} & -\frac{c_x^1 - c_x^2}{c^1 - c^2} \\ \frac{c_x^1 - c_x^2}{c^1 - c^2} & c_x^2 - \frac{c_x^1 - c_x^2}{c^1 - c^2} \end{pmatrix} \partial_x^{-1}. \tag{24}$$

**Example 2** *Let us consider equation (6) with  $m = 2$ . The equation becomes*

$$\begin{aligned} c_t^1 &= c^1 c_x^1 + 2c_x^1 - c_x^2 - c_x^3 \\ c_t^2 &= c^2 c_x^2 + 2c_x^1 - c_x^2 - c_x^3 \\ c_t^3 &= c^3 c_x^3 + 2c_x^1 - c_x^2 - c_x^3 \end{aligned} \tag{25}$$

*The above system admits the recursion operator*

$$\begin{pmatrix} c_x^1 + \frac{c_x^1 - c_x^2}{c^1 - c^2} + \frac{c_x^1 - c_x^3}{c^1 - c^3} & -\frac{c_x^1 - c_x^2}{c^1 - c^2} & -\frac{c_x^1 - c_x^3}{c^1 - c^3} \\ 2\frac{c_x^1 - c_x^2}{c^1 - c^2} & c_x^2 - 2\frac{c_x^1 - c_x^2}{c^1 - c^2} + \frac{c_x^2 - c_x^3}{c^2 - c^3} & -\frac{c_x^2 - c_x^3}{c^2 - c^3} \\ 2\frac{c_x^1 - c_x^3}{c^1 - c^3} & -\frac{c_x^3 - c_x^2}{c^3 - c^2} & c_x^3 - 2\frac{c_x^1 - c_x^3}{c^1 - c^3} + \frac{c_x^3 - c_x^2}{c^3 - c^2} \end{pmatrix} \partial_x^{-1}. \tag{26}$$

**Example 3** *Let us consider equation (6) with  $m = 3$ . The equation becomes*

$$\begin{aligned} c_t^1 &= c^1 c_x^1 + 3c_x^1 - c_x^2 - c_x^3 - c_x^4 \\ c_t^2 &= c^2 c_x^2 + 3c_x^1 - c_x^2 - c_x^3 - c_x^4 \\ c_t^3 &= c^3 c_x^3 + 3c_x^1 - c_x^2 - c_x^3 - c_x^4 \\ c_t^4 &= c^4 c_x^4 + 3c_x^1 - c_x^2 - c_x^3 - c_x^4 \end{aligned} \tag{27}$$



The above system admits the recursion operator

$$\left( \begin{array}{cccc} c_x^1 & -\frac{c_x^1-c_x^2}{c^1-c^2} & -\frac{c_x^1-c_x^3}{c^1-c^3} & -\frac{c_x^1-c_x^4}{c^1-c^4} \\ 3\frac{c_x^1-c_x^2}{c^1-c^2} & c_x^2 - 3\frac{c_x^1-c_x^2}{c^1-c^2} & -\frac{c_x^2-c_x^3}{c^2-c^3} & -\frac{c_x^2-c_x^4}{c^2-c^4} \\ 3\frac{c_x^1-c_x^3}{c^1-c^3} & -\frac{c_x^3-c_x^2}{c^3-c^2} & c_x^3 - 3\frac{c_x^1-c_x^3}{c^1-c^3} & -\frac{c_x^3-c_x^4}{c^3-c^4} \\ 3\frac{c_x^1-c_x^4}{c^1-c^4} & -\frac{c_x^4-c_x^2}{c^4-c^2} & -\frac{c_x^4-c_x^3}{c^4-c^3} & c_x^4 - 3\frac{c_x^4-c_x^1}{c^4-c^1} \end{array} \right) \partial_x^{-1} + . \quad (28)$$

$$\left( \begin{array}{cccc} \sum_{j=2}^4 \frac{c_x^1-c_x^j}{c^1-c^j} & 0 & 0 & 0 \\ 0 & \sum_{j=2, j \neq 2}^4 \frac{c_x^2-c_x^j}{c^2-c^j} & 0 & 0 \\ 0 & 0 & \sum_{j=2, j \neq 3}^4 \frac{c_x^3-c_x^j}{c^3-c^j} & 0 \\ 0 & 0 & 0 & \sum_{j=2, j \neq 4}^4 \frac{c_x^4-c_x^j}{c^4-c^j} \end{array} \right) \partial_x^{-1}$$

## References

- [1] M. Gürses, A. Karasu, V.V. Sokolov, " On construction of recursion operator from Lax representation", *J. Math. Phys*, **40**, no. 12 (1999) 6473-6490
- [2] M. Blaszak, On the construction of recursion operator and algebra of symmetries for field and lattice systems. Proceedings of the XXXII Symposium on Mathematical Physics (Torun, 2000). Rep. Math. Phys. 48 , no. 1-2, (2001) 27-38.
- [3] Gürses M., Zheltukhin K. Recursion operators of some equations of hydrodynamic type. J. Math. Phys. 42 , no. 3, (2001).
- [4] Zheltukhin K. Recursion operator and dispersionless rational Lax representation. Phys. Lett. A 297 , no. 5-6, (2002).

- [5] A.P. Fordy and B. Gürel, "A new construction of recursion operators for systems of hydrodynamic type", *Theoret. Math. Phys.*, 122, no. 1, (2000) 29-38.
- [6] M.B. Sheftel, "Generalized Hydrodynamic-type systems" in *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 3. Edited by N.H. Ibragimov, p.169-189. CRC Press, New York, 1996.
- [7] V.M. Teshukov, "Hyperbolic systems admitting a non-trivial Lie-Backlund group", *LIIAN*, **106**, (1989) 25-30.
- [8] I.M. Gel'fand, L.A. Dikii, Asymptotic Behaviour of the Re-solvent of Sturm-Liouville equations and the Algebra of the Korteweg-de Vries equations, *Funk. Anal. Appl.*, **10**, no. 13 (1976).
- [9] B.A. Dubrovin geometry of 2D Topological Field Theories, *Lecture Notes in Mathematics*, vol. 1620, Springer Berlin, (1993)
- [10] L. Chierchia, N. Ercolani, D. W. McLaughlin, On the weak limit of rapidly oscillating waves, *Duke Math. J.*, 55 (1987) 759-764.
- [11] B. A. Dubrovin, S. P. Novikov, Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, *Soviet Math. Dokl.*, 27 (1983) 781-785.
- [12] H. Flaschka, M.G. Forest, D.W. McLaughlin, Multiphase averaging and the inverse spectral solution of the Korteweg-de Vries equation, *Comm. Pure Appl. Math.*, 33 (1980)
- [13] I. M. Krichever, The averaging method for two-dimensional integrable equations, *Funct. Anal. Appl.*, 22, No. 3 (1988) 200-213.
- [14] P.D. Lax and C.D. Levermore, The Small Dispersion Limit for the Korteweg-de Vries Equation I, II, III, *Comm. Pure Appl. Math.*, 36 (1983) 253-290.

- [15] S.P. Tsarev, On the integrability of the averaged KdV and Benney equations, Proc. NATO ARW Singular Limits and Dispersive Waves, Lyon, France, 8-12 July, 1991.
- [16] S.P. Tsarev, Classical differential geometry and integrability of systems of hydro-dynamic type. In: Proc. NATO ARW Applications of Analytic and Geometrical Methods to Nonlinear Differential Equations, 14-19 July, 1992, Exeter, UK, NATO ASI Series C 413, (ed. P.A.Clarkson), Kluwer Publ., 241-249.
- [17] S.P. Tsarev, Integrability of equations of hydrodynamic type from the end of the 19th to the end of the 20th century. In: Integrability: the Seiberg-Witten and Whitham equations (Edinburgh, 1998) p. 251-265, Gordon and Breach, Amsterdam, 2000.
- [18] B. Szablikowski, M. Blaszk, Meromorphic Lax representations of  $(1+1)$ -dimensional multi-Hamiltonian dispersionless systems. J. Math. Phys. 47 , no. 9, (2006) 092701.
- [19] Pavlov M. V., The Kupershmidt hydrodynamic chains and lattices, International Mathematics Research Notices( IMRN) , Vol. 2006, 1-43.